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The Quantization of Fermi Systems and the Dirac Bracket

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Abstract

This work is devoted (a) to discussing some problems related to the quantization rule for constrained classical models of Fermi type, and (b) to working with some detail a specific model which is a classical analogue of the quantum Fermi systems. The quantization of this model is shown to depend on the addition of a total time derivative to the corresponding Lagrangian.

1. *Introduction*

1.1. *General*

The basic purpose of this work is twofold: (i) to discuss some problems concerning the quantization rules of classical systems, and (ii) to look for a classical analogue of the quantum Fermi systems in a sense that will be specified below.

Let us remember that there exists a clear asymmetry in the standard exposition of the quantization procedure for ordinary (unconstrained) classical systems, for besides the quantization rule

$$i\{,\}_- \to [,]_- \tag{1.1.1}$$

valid only for integer-spin (Bose) systems, a corresponding one valid for half integer-spin (Fermi) systems is not presented. An important step towards the treatment of both Bose and Fermi systems on an equal footing has been given by Droz-Vincent (1966), who has shown that, besides the usual (skew-symmetric) algebraic structure, there exists another, symmetric,

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structure for ordinary classical systems. This structure is characterized by the existence of a new bracket, $\{,\}_+$ (called a 'plus Poisson bracket') which is the symmetric partner of the usual (minus) Poisson bracket [see (2.2)]. This suggests setting up the rule

$$\xi\{,\}_+ \rightarrow [,]_+ \tag{1.1.2}$$

which is the counterpart of (1.1.1) valid now for the quantization of the (eventually existing) unconstrained Fermi-like classical models. ξ is a parameter of the theory whose values will be discussed below [see also Franke & Kálnay (1970)].†

The quantization procedures based in (1.1.1) and (1.1.2) must be modified to encompass an important class of classical problems for which, as Dirac (1950, 1951, 1958, 1964) showed, the ordinary canonical formalism fails. For these systems the canonical coordinates and momenta are not independent, but are bound to satisfy a certain type of constrains $\theta^a(q,p)$

$$\theta^a \approx 0 \tag{1.1.3}$$

called second-class constraints.‡ The second-class constraints form a subset of the whole set of constraints which may exist in the problem. They are characterized by the existence of the matrix $\|C_{ab}^-\|$ defined by

$$C_{ab}^-\{\theta^b, \theta^c\}_- \equiv \delta_{ac} \tag{1.1.4}$$

Dirac has shown that for these systems (called here 'constrained systems') the quantization rule (1.1.1) is not consistent and must be modified by the introduction of a new bracket (called a 'Dirac bracket'), which has the form

$$\{F, G\}_-^* \equiv \{F, G\}_- - \{F, \theta^a\}_- C_{ab}^-\{\theta^b, G\}_- \tag{1.1.5}$$

In terms of it, the quantization rule for constrained Bose systems reads

$$i\{,\}_-^* \rightarrow [,]_- \tag{1.1.6}$$

Recently, the above considerations have been extended in order to treat constrained Fermi-like systems (Francke & Kálnay, 1970). This was done by showing the existence of a dual symmetric partner of the (minus) Dirac bracket, (1.1.5). This bracket (called a 'plus Dirac bracket'), is given by

$$\{F, G\}_+^* \equiv \{F, G\}_+ - \{F, \theta^a\}_+ C_{ab}^+\{\theta^b, G\}_+ \tag{1.1.7}$$

Here, the set of *plus* second-class constraints θ^a is such that the matrix $\|C_{ab}^+\|$, inverse to the matrix of the plus Poisson brackets between the θ's, exists [cf. equation (1.1.4)]§

† They have set $\xi = i$. However, see below.

‡ The weak equality sign $\approx$, introduced by Dirac, means that equality to zero can be used in an equation only after all brackets (or partial differentiations) have been calculated.

§ For simplicity we use the same symbol θ to denote a plus second-class constraint or a minus second-class constraint. However, notice that both subsets of constraints are usually different [see equations (1.1.4) and (1.1.8)].

$$C^{+}_{ab}\{\theta^b, \theta^c\}_+ \equiv \delta_{ac} \tag{1.1.8}$$

Constrained classical Fermi-like systems must then be quantized according to

$$\xi\{,\}_+{}^* \to [,]_+ \tag{1.1.9}$$

Relations (1.1.2) and (1.1.9) close the previously existing gap regarding the quantization of classical systems. The resulting scheme is valid for both ordinary or constrained systems of Bose or Fermi type.

1.2. *The Parameter* ξ

As we have shown above, the quantization problem for both Bose and Fermi systems can be formally treated in a very symmetric way. Nevertheless, there exists an important difference between the two cases. To ascertain where this difference lies, let us consider first the rules (1.1.1) and (1.1.6). The stronger argument in favour of the hypothesis of the universal validity of (1.1.1) is, of course, its capability of verification for all systems for which both the (unconstrained) classical and quantum Bose models are separately known to be correct descriptions of the respective régimes. With regard to (1.1.6), it is important in our context to note that, as the Dirac bracket is the natural generalization for constrained systems of the Poisson bracket, taking the same i factor in (1.1.6) as in (1.1.1) it amounts to do an unicity hypothesis concerning the quantization rules.† An identical hypothesis has been made when writing the same parameter ξ in (1.1.9) as in (1.1.2), but, as we are going to see, some other elements must be considered.

The whole quantization problem for Fermi systems is of a somewhat different nature when we realize that the quantization of the classical models of some well-known Fermi systems (e.g. a plasma) has not been studied yet in connexion with quantization rules like (1.1.2) or (1.1.9). Consequently, a verification of (1.1.2), for example, has not been made. Although such verification can be in principle undertaken, to do it is not, however, the purpose of this work; instead, we are interested here in some problems related to the theoretical consistence of the quantization scheme. We are interested particularly in the possibility of constructing classical analogues, i.e., classical models which formally reproduce the known (quantum) Fermi systems when the rules (1.1.2) or (1.1.9) are used. Turning to the parameter ξ, we see that if we had studied an unconstrained classical analogue of some Fermi system, then the (real) value, or values, of it could be determined. The rules (1.1.2) and (1.1.9) would then be fully specified. A reason why we are interested in leaving open the values of ξ is that, as far as we know, such classical analogue has not been studied (or even its existence proved). Furthermore, we are going to consider in Section 3 a classical model which corresponds, via the rule (1.1.9), to the second

† This hypothesis is not a weak one because the complex character of the Dirac bracket (1.1.5) depends in general on the constraints. We are then, in fact, restricting the class of constraints (and then of Lagrangians) which are to be present in the classical models of Bose systems.

quantized Fermi systems, *for all values of* $\xi \neq 0$ *in the complex domain.* In this model the set of values ξ can take is in correspondence with Lagrangians which differ from each other by a total time derivative. If we make the hypothesis of a unique value of ξ, then all these 'equivalent' Lagrangians give rise, in fact, to different quantized systems. We may try to avoid this considering, for example, that there exists an intrinsic ambiguity in the rule (1.1.9) which permits all values of $\xi \neq 0$. Even in this case, we shall be led below to the conclusion that the addition of a total time derivative to our Lagrangian is relevant to the quantization scheme for Fermi-like systems.

2. *Notations and Conventions*

The sum convention is used in any place as well as $\hbar \neq 1$ unless the contrary is explicitly stated.

The coordinates of the classical model to be worked out below: a_r, $r = 1, 2, \ldots, N$, and its complex conjugated $\bar{a}_r$, are concisely denoted by a_{Ar} with

$$a_{Ar} \equiv \delta_{A\mathrm{I}}\, a_r + \delta_{A\mathrm{II}}\, \bar{a}_r \tag{2.1}$$

Here and elsewhere, the capital index takes the values I and II. The corresponding canonically conjugate momenta are p_{Ar}, but $p_{\mathrm{II}r}$ is not necessarily equal to $\bar{p}_{\mathrm{I}r}$.

Brackets. Commutators and anticommutators are denoted by square brackets: $[,]_\pm$. Curly brackets indicate (ordinary) minus Poisson brackets and plus Poisson brackets:

$$\{F, G\}_\pm \equiv \{F(a,p), G(a,p)\}_\pm \equiv \frac{\partial F}{\partial a_{Ar}} \frac{\partial G}{\partial p_{Ar}} \pm \frac{\partial F}{\partial p_{Ar}} \frac{\partial G}{\partial a_{Ar}} \tag{2.2}$$

where

$$a \equiv (a_{\mathrm{I}1}, \ldots, a_{\mathrm{I}N}, a_{\mathrm{II}1}, \ldots, a_{\mathrm{II}N})$$
$$p \equiv (p_{\mathrm{I}1}, \ldots, p_{\mathrm{I}N}, p_{\mathrm{II}1}, \ldots, p_{\mathrm{II}N})$$

Curly brackets with an asterisk denote plus and minus (ordinary) Dirac brackets: $\{,\}_\pm^*$. [They are defined by (1.1.5) and (1.1.7).]

Matrices. σ_{AB} and $\sigma'_{AB}(\xi')$ are the elements of the 2×2 matrices

$$\|\sigma_{AB}\| \equiv \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \quad \text{and} \quad \|\sigma'_{AB}(\xi')\| \equiv \frac{1}{2} \begin{Vmatrix} 0 & \xi' - i \\ \xi' + i & 0 \end{Vmatrix} \tag{2.3}$$

ξ' is an arbitrary complex parameter.

Operators are denoted by symbols with a circumflex, and a dagger means Hermitian conjugation.

3. *A Classical Analogue of Fermi Systems*

Let us consider the model Lagrangian

$$L_{\xi'}(a, \dot{a}) = \dot{a}_{Ar}\, \sigma'_{AB}(\xi')\, a_{Br} - \tfrac{1}{2}\omega_r\, a_{Ar}\, \sigma_{AB}\, a_{Br} - \mathscr{U}(a) \tag{3.1}$$

where the ω's are some characteristic constants and $\mathscr{U}(a)$ is a real differentiable function of the coordinates.

The canonical momenta and the Hamiltonian derived from (3.1) are

$$p_{Ar} = \sigma'_{AB}(\xi')\, a_{Br} \tag{3.2}$$

and

$$H = \tfrac{1}{2}\omega_r\, a_{Ar}\, \sigma_{AB}\, a_{Br} + \mathscr{U}(a) \tag{3.3}$$

The canonical coordinates and momenta are then not independent but satisfy the primary constraints

$$\phi^{Ar}_{\xi'} \equiv p_{Ar} - \sigma'_{AB}(\xi')\, a_{Br} \approx 0 \tag{3.4}$$

To find the canonical equations we must then resort to Dirac's generalization of the ordinary Hamiltonian theory (Dirac, 1950, 1951, 1958, 1964). The corresponding equations are

$$\dot{a}_{Ar} = \frac{\partial H}{\partial p_{Ar}} + u_{Bs}\frac{\partial \phi^{Bs}_{\xi'}}{\partial p_{Ar}} \tag{3.5a}$$

$$\dot{p}_{Ar} = -\frac{\partial H}{\partial a_{Ar}} - u_{Bs}\frac{\partial \phi^{Bs}_{\xi'}}{\partial a_{Ar}} \tag{3.5b}$$

where the u_{Bs}'s are non-canonical variables which take into account the existence of the constraints (3.4). As the Hamiltonian (3.3) does not depend on the momenta, and as the constraints are linear in them, it follows from (3.5a) that

$$u_{Bs} = \dot{a}_{Bs} \tag{3.6}$$

This result was to be expected, because the defining relations (3.2) do not permit the obtaining of the velocities in terms of the canonical coordinates and momenta. These velocities then appear as additional variables in the formalism. By substitution of (3.6) in (3.5b) we are led to

$$\dot{p}_{Ar} = -\omega_r\, \sigma_{AB}\, a_{Br} + \dot{a}_{Br}\, \sigma'_{BA}(\xi') - \frac{\partial \mathscr{U}}{\partial a_{Ar}} \tag{3.7}$$

which, after using (3.2), becomes the following differential equations for the coordinates

$$(\sigma'_{BA}(\xi') - \sigma'_{AB}(\xi'))\, \dot{a}_{Br} = \omega_r\, \sigma_{AB}\, a_{Br} + \frac{\partial \mathscr{U}}{\partial a_{AR}} \tag{3.8}$$

These equations are equivalent to

$$\dot{a}_r = i\omega_r\, a_r + i\frac{\partial \mathscr{U}}{\partial \bar{a}_r} \tag{3.9}$$

and the one obtained from it by complex conjugation.†

It follows from (3.3) and (3.9) that our model represents nothing but a

† Of course, equations (3.8) and (3.9) can also be obtained as the Euler-Lagrange equations of (3.1), but, as pointed out by Dirac, the canonical formalism is the best starting-point to quantization. In equations (3.7)–(3.9) the index r is not a dummy summation index.

standard system of harmonic oscillators of frequencies ω_r and coupling energy $\mathscr{U}$. What is not standard is the form (3.1) of the Lagrangian, the first term of which is responsible for the existence of the constraints. These constraints play an essential role in the process of quantization. In general, additional, secondary constraints may arise as a consequence of the fact that the primary ones must hold for any time, i.e. they must satisfy $\dot{\phi}_{\xi'}^{Ar} \approx 0$. As the $\phi_{\xi'}^{Ar}$ are linear in the momenta, the manipulations made above show at once that the consistence equations will reduce in fact to (3.6). In other words, there are no secondary constrains in our problem.

Let us now consider the quantization problem. For this it is crucial what class the constraints (3.4) are. As we are primarily interested in the possibility of quantizing the model (3.1) according to a symmetric rule, the plus Poisson bracket between the $\phi_{\xi'}^{Ar}$ must be calculated. They are easily found from (3.4) as

$$\{\phi_{\xi'}^{Ar}, \phi_{\xi'}^{Bs}\}_+ = -\delta_{rs}(\sigma'_{AB}(\xi') + \sigma'_{BA}(\xi')) = -\xi'\,\delta_{rs}\,\sigma_{AB} \tag{3.10}$$

The constraints are then first class if $\xi' = 0$ and second class otherwise. Let us take for the moment the later case. When the constraints are second class the quantization must proceed by determining the algebra of classical variables as defined by the symmetric Dirac bracket relations. We now calculate them for two general dynamical variables: $F(a,p)$ and $G(a,p)$.

Note first that in virtue of (3.10) the matrix elements $C^+_{Ar,Bs}$ are simply

$$C^+_{Ar,Bs}(\xi') = -\xi'^{-1}\,\delta_{rs}\,\sigma_{AB} \tag{3.11}$$

Furthermore, we have

$$\{F, \phi_{\xi'}^{Bs}\}_+ = \frac{\partial F}{\partial a_{Bs}} - \sigma'_{BA}(\xi')\frac{\partial F}{\partial p_{As}} \tag{3.12}$$

and the corresponding expression for G. Combining (1.1.7), (3.11) and (3.12) we are led to

$$\{F, G\}_+^* = \{F, G\}_+ + \xi'^{-1}\sum_r \left(\frac{\partial F}{\partial a_{Br}} - \sigma'_{BA}(\xi')\frac{\partial F}{\partial p_{Ar}}\right)\sigma_{BD} \times \left(\frac{\partial G}{\partial a_{Dr}} - \sigma'_{DC}(\xi')\frac{\partial G}{\partial p_{Cr}}\right) \tag{3.13}$$

or, in a more explicit form, to

$$\{F, G\}_+^* = \{F, G\}_+ + \xi'^{-1}\sum_r \left(\frac{\partial F}{\partial a_{\mathrm{I}r}} - \tfrac{1}{2}(\xi' - i)\frac{\partial F}{\partial p_{\mathrm{II}r}}\right) \times \left(\frac{\partial G}{\partial a_{\mathrm{II}r}} - \tfrac{1}{2}(\xi' + i)\frac{\partial G}{\partial p_{\mathrm{I}r}}\right) + \left(\frac{\partial F}{\partial a_{\mathrm{II}r}} - \tfrac{1}{2}(\xi' + i)\frac{\partial F}{\partial p_{\mathrm{I}r}}\right) \times \left(\frac{\partial G}{\partial a_{\mathrm{I}r}} - \tfrac{1}{2}(\xi' - i)\frac{\partial G}{\partial p_{\mathrm{II}r}}\right) \tag{3.14}$$

(3.13) or (3.14) is the general Dirac bracket relation we looked for. It is known, however, that the standard rule of quantization (1.1.1) is consistent only when applied to the elementary variables of the theory, i.e., canonical coordinates and momenta (Peirls, 1952; Bergmann & Goldberg, 1955). The same happens with (1.1.6). Thus we only need here the brackets between the a's and p's which from (3.13) take the form:

$$\{a_{Ar}, a_{Bs}\}_+{}^* = \xi'^{-1}\,\delta_{rs}\,\sigma_{AB} \tag{3.15a}$$

$$\{a_{Ar}, p_{Bs}\}_+{}^* = \xi'^{-1}\,\delta_{rs}\,\sigma_{AC}\,\sigma'_{BC}(\xi') \tag{3.15b}$$

$$\{p_{Ar}, p_{Bs}\}_+{}^* = \xi'^{-1}\,\delta_{rs}\,\sigma'_{CA}(\xi')\,\sigma_{CD}\,\sigma'_{DB}(\xi') \tag{3.15c}$$

It follows from this that if we quantize our classical model according to (1.1.9), setting $\xi = \xi'$, the quantum a-variables that are such that

$$\begin{aligned} a_{\mathrm{I}r} &\equiv a_r \to \hat{a}_r \\ a_{\mathrm{II}r} &\equiv \bar{a}_r \to \hat{a}_r{}^\dagger \end{aligned} \tag{3.16a}$$

will satisfy the commutation relations

$$[\hat{a}_r, \hat{a}_s{}^\dagger]_+ = \delta_{rs}; \qquad [\hat{a}_r, \hat{a}_s]_+ = [\hat{a}_r{}^\dagger, \hat{a}_s{}^\dagger]_+ = 0 \tag{3.17}$$

These are characteristics of the creation and annihilation operators of a Fermi system, the states of which are labeled by the index r. According to this (3.3) and (3.9) we see that the model we have worked out is a classical analogue of the well-known (second quantized) systems of Fermions.

The model we are considering is then an example of a quantization process via the rule (1.1.9) which is well defined up to an arbitrary complex factor $\xi = \xi' \neq 0$. As can be seen from (3.1), the different values of ξ' correspond to Lagrangians which differ by a total time derivative.† This way we arrive at the (real) Lagrangian corresponding to the singular value $\xi' = 0$. In this case the Dirac bracket is not defined because, as it was found above, the constraints (3.2) are all first class. The only symmetric rule of quantization we have then at hand is (1.1.2). Nevertheless, *this rule gives bracket relations essentially different to* (3.15). Now, a quantized system is defined not only by its equations of motion but also by the algebra of its canonical variables. The above discussion then shows that two Lagrangians which differ from each other by a total time derivative may give rise to two different quantized systems‡.

It is worth to note finally that our model can also be quantized according to the skew-symmetric rule (1.1.6). The problem is simpler here because a calculation, similar to that made above, shows that irrespective of the value of ξ', we find (a) the constraints are always second class, and, (b) the quantization rule is uniquely defined and it coincides with (1.1.6). The question arises if in more general cases changing the Lagrangian by a total time derivative may produce also essential changes in the quantum operators algebra derived from the skew-symmetric quantization rule (1.1.6)§.

† Note, however, that the Hamiltonian (3.3) does not depend on ξ'.

‡ A strictly parallel situation is found when quantizing the Dirac Field (Kálnay, 1971).

§ Kálnay, A. J. and Ruggeri, G. J. *Gauge-Variance of the Dirac Brackets.* (To be submitted for publication.)

4. *Summary*

The main results of this work are twofold (a) a constrained classical analogue of Fermi systems has been constructed,† and (b) in the light of this model at least two general questions related to the quantization rules for Fermi-like systems arise, which require further investigation. These questions are connected with, on the one hand, the eventual ambiguity of these rules, and on the other with its sensitivity to the change of the Lagrangian by a total time derivative. If the quantization scheme is taken for granted we are led to conclude that two Lagrangians which differ from each other by a total time derivative define, in general, different quantized systems.

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† The classical limit of para-Fermi variables is considered by Kálnay (1971).